Lecture 12 - Model-Free Policy Evaluation

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DDA4230: Reinforcement Learning
Course Page: [Click]

Monte-Carlo Policy Evaluation

An example of the Monte-Carlo method: Suppose we want to estimate how long the commute from your house to the campus will take today.

- We have access to a commute simulator that models our uncertainty of how bad the traffic will be, the weather, construction delays, and other variables, as well as how these variables interact with each other.
- We estimate the expected commute time by simulating our commute many times on the simulator and then take an average over the simulated commute times.

This is called a Monte-Carlo estimate of our commute time. Monte-Carlo method only works in episodic environments

Monte-Carlo Policy Evaluation

In the context of reinforcement learning, the quantity we want to estimate is $V^{\pi}(s)$, which is the average of returns G_t (which equals R_t without n-step truncate or eligibility traces) under policy π starting at state s. We can thus get a Monte-Carlo estimate of $V^{\pi}(s)$ through three steps:

1. Execute a rollout of policy π until termination many times;
2. Record the returns G_t that we observe when starting at state s;
3. Take an average of the values we get for G_t to estimate $V^{\pi}(s)$.

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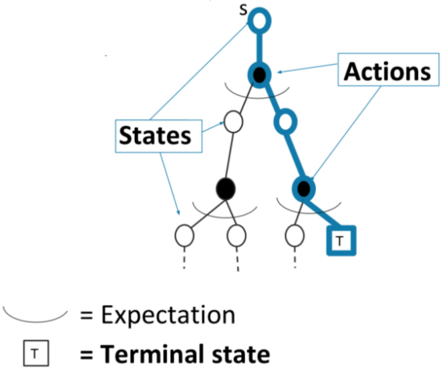
Monte-Carlo Policy Evaluation



The backup diagram for Monte-Carlo policy evaluation. The new blue line indicates that we sample an entire episode until termination starting at state s.

Monte-Carlo Policy Evaluation

First-visit Monte-Carlo: Take an average over just the first time we visit a state in each rollout.

Algorithm 1: First-visit Monte-Carlo policy evaluation

Input: $h_1, \ldots, h_j$

For all states s, $N(s) \leftarrow 0$, $S(s) \leftarrow 0$, $V(s) \leftarrow 0$

for *each episode* h_j **do**

  for $t = 1, \ldots, L_j$ **do**

    if $s_{j,t} \neq s_{j,u}$ *for* $u < t$ **then**

      $N(s_{j,t}) \leftarrow N(s_{j,t}) + 1$

      $S(s_{j,t}) \leftarrow S(s_{j,t}) + G_{j,t}$

      $V^\pi(s_{j,t}) \leftarrow S(s_{j,t})/N(s_{j,t})$

return V^π

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Monte-Carlo Policy Evaluation

Every-visit Monte-Carlo: Take an average over every time we visit the state in each rollout. If we are in a truly Markovian-domain, every-visit Monte Carlo will be more data efficient because we update our average return for a state every time we visit the state.

Algorithm 2: Every-visit Monte-Carlo policy evaluation

Input: $h_1, \ldots, h_j$
For all states s, $N(s) \leftarrow 0$, $S(s) \leftarrow 0$, $V(s) \leftarrow 0$
for *each episode* h_j **do**
  for $t = 1, \ldots, L_j$ **do**
    $N(s_{j,t}) \leftarrow N(s_{j,t}) + 1$
    $S(s_{j,t}) \leftarrow S(s_{j,t}) + G_{j,t}$
    $V^\pi(s_{j,t}) \leftarrow S(s_{j,t}) / N(s_{j,t})$
return V^π

Monte-Carlo Policy Evaluation

In these Algorithms, we can remove vector S and replace the update for $V^\pi(s_{j,t})$ with

$$V^\pi(s_{j,t}) \leftarrow V^\pi(s_{j,t}) + \frac{1}{N(s_{j,t})}(G_{j,t} - V^\pi(s_{j,t})).$$

This is because the new average is the average of $N(s_{j,t}) - 1$ of the old values $V^\pi(s_{j,t})$ and the new return $G_{j,t}$, giving us

$$\frac{V^\pi(s_{j,t}) \cdot (N(s_{j,t}) - 1) + G_{j,t}}{N(s_{j,t})} = V^\pi(s_{j,t}) + \frac{1}{N(s_{j,t})}(G_{j,t} - V^\pi(s_{j,t})),$$

Replacing $1/N(s_{j,t})$ with α in this new update gives us the more general incremental Monte-Carlo policy evaluation.

Monte-Carlo Policy Evaluation

Incremental First-visit Monte-Carlo policy evaluation:

Algorithm 3: Incremental first-visit Monte-Carlo policy evaluation

Input: $\alpha, h_1, \ldots, h_j$

For all states s, $N(s) \leftarrow 0$, $V(s) \leftarrow 0$

for *each episode* h_j **do**

  for $t = 1, \ldots, terminal$ **do**

    if $s_{j,t} \neq s_{j,u}$ *for* $u < t$ **then**

      $N(s_{j,t}) \leftarrow N(s_{j,t}) + 1$

      $V^\pi(s_{j,t}) \leftarrow V^\pi(s) + \alpha(G_{j,t} - V^\pi(s))$

return V^π

Monte-Carlo Policy Evaluation

Incremental Every-visit Monte-Carlo policy evaluation:

Algorithm 4: Incremental every-visit Monte-Carlo policy evaluation

Input: $\alpha, h_1, \dots, h_j$
For all states s, $N(s) \leftarrow 0$, $V(s) \leftarrow 0$
for *each episode* h_j **do**
  for $t = 1, \dots, terminal$ **do**
    $N(s_{j,t}) \leftarrow N(s_{j,t}) + 1$
    $V^\pi(s_{j,t}) \leftarrow V^\pi(s) + \alpha(G_{j,t} - V^\pi(s))$
return V^π

Setting $\alpha = 1/N(s_{j,t})$ recovers the original Monte-Carlo policy evaluation algorithms given in the above Algorithms, while setting $\alpha > \frac{1}{N(s)}$ gives a higher weight to newer data, which can help learning in non-stationary domains.

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Monte-Carlo Off-Policy Evaluation

Motivation:

- In the above, we discussed the case where we are able to obtain many realizations of G_t under the policy π that we want to evaluate.
- However, in many costly or high-risk situations, we are unable to obtain rollouts of G_t under the policy that we wish to evaluate.
- In this section, we describe Monte-Carlo off-policy policy evaluation, a method for using data from one policy to evaluate a different policy.

Monte-Carlo Off-Policy Evaluation

Importance Sampling: that estimates the expected value of a function $f(x)$ when x is drawn from the distribution q using only the data $f(x_1), \ldots, f(x_n)$, where x_i are drawn from a different distribution p. In summary, given $q(x_i), p(x_i), f(x_i)$ for $1 \leq x_i \leq n$, we would like an estimate for $\mathbb{E}_{x\sim q}[f(x)]$. We can do this via the approximation:

$$
\begin{aligned}
\mathbb{E}_{x\sim q}[f(x)] &= \int_x q(x)f(x)dx \\
&= \int_x p(x)\left[\frac{q(x)}{p(x)}f(x)\right]dx \\
&= \mathbb{E}_{x\sim p}\left[\frac{q(x)}{p(x)}f(x)\right] \\
&\approx \frac{1}{n}\sum_{i=1}^{n}\left[\frac{q(x_i)}{p(x_i)}f(x_i)\right].
\end{aligned}
$$

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Monte-Carlo Off-Policy Evaluation

Importance sampling for off-policy policy evaluation: We apply importance sampling estimates to reinforcement learning. In this instance, we want to approximate the value of state s under policy π_1, given by $V^{\pi_1}(s) = \mathbb{E}[G_t \mid s_t = s]$, using n histories $h_1, \ldots, h_n$ generated under policy π_2. The importance sampling estimate result provides:

$$V^{\pi_1}(s) \approx \frac{1}{n} \sum_{j=1}^{n} \frac{\mathbb{P}(h_j \mid \pi_1, s)}{\mathbb{P}(h_j \mid \pi_2, s)} G(h_j),$$

where $G(h_j) = \sum_{t=1}^{L_j - 1} \gamma^{t-1} r_{j,t}$ is the total discounted sum of rewards for history h_j.

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Monte-Carlo Off-Policy Evaluation

Now, for a general policy π, we have that the probability of experiencing history h_j under policy π is

$$\mathbb{P}(h_j \mid \pi, s = s_{j,1}) = \prod_{t=1}^{L_j - 1} \mathbb{P}(a_{j,t} \mid s_{j,t}) \mathbb{P}(r_{j,t} \mid s_{j,t}, a_{j,t}) \mathbb{P}(s_{j,t+1} \mid s_{j,t}, a_{j,t})$$

where L_j is the length of the j-th episode. In each transition, the components are 1) $\mathbb{P}(a_{j,t} \mid s_{j,t})$ - probability we take action $a_{j,t}$ at state $s_{j,t}$; 2) $\mathbb{P}(r_{j,t} \mid s_{j,t}, a_{j,t})$ - probability we experience reward $r_{j,t}$ after taking action $a_{j,t}$ in state $s_{j,t}$; 3) $\mathbb{P}(s_{j,t+1} \mid s_{j,t}, a_{j,t})$ - probability we transition to state $s_{j,t+1}$ after taking action $a_{j,t}$ in state $s_{j,t}$.

Monte-Carlo Off-Policy Evaluation

Combining our importance sampling estimate for $V^{\pi_1}(s)$ with our decomposition of the history probabilities, $\mathbb{P}(h_j \mid \pi, s = s_{j,1})$, we get that

$$
\begin{aligned}
V^{\pi_1}(s) &\approx \frac{1}{n}\sum_{j=1}^{n} \frac{\mathbb{P}(h_j \mid \pi_1, s)}{\mathbb{P}(h_j \mid \pi_2, s)} G(h_j) \\
&= \frac{1}{n}\sum_{j=1}^{n} \frac{\prod_{t=1}^{L_j-1} \pi_1(a_{j,t} \mid s_{j,t})\mathbb{P}(r_{j,t} \mid s_{j,t}, a_{j,t})\mathbb{P}(s_{j,t+1} \mid s_{j,t}, a_{j,t})}{\prod_{t=1}^{L_j-1} \pi_2(a_{j,t} \mid s_{j,t})\mathbb{P}(r_{j,t} \mid s_{j,t}, a_{j,t})\mathbb{P}(s_{j,t+1} \mid s_{j,t}, a_{j,t})} G(h_j) \\
&= \frac{1}{n}\sum_{j=1}^{n} G(h_j) \prod_{t=1}^{L_j-1} \frac{\pi_1(a_{j,t} \mid s_{j,t})}{\pi_2(a_{j,t} \mid s_{j,t})}.
\end{aligned}
$$

Temporal Difference Learning

Motivation. A recap of the policy evaluation methods:

- **Dynamic programming** leverages bootstrapping to help us get value estimates with only one backup.
- **Monte Carlo** samples many histories for many trajectories which frees us from using a model.
- **Temporal difference learning** combines bootstrapping with sampling to give us a new model-free policy evaluation algorithm.

Temporal Difference Learning

To see how to combine sampling with bootstrapping, we go back to our incremental Monte-Carlo update

$$V^{\pi}(s_t) \leftarrow V^{\pi}(s_t) + \alpha(G_t - V^{\pi}(s_t)).$$

We replace G_t with a Bellman backup like $r_t + \gamma V^{\pi}(s_{t+1})$, where r_t is a sample of the reward at time step t and $V^{\pi}(s_{t+1})$ is our current estimate of the value at the next state. It gives us the temporal difference (TD) learning update

$$V^{\pi}(s_t) \leftarrow V^{\pi}(s_t) + \alpha(r_t + \gamma V^{\pi}(s_{t+1}) - V^{\pi}(s_t)).$$

Temporal Difference Learning

The TD error is given by:

$$\delta_t = r_t + \gamma V^\pi(s_{t+1}) - V^\pi(s_t)$$

The sampled reward combined with the bootstrap estimate of the next state value, i.e., the TD target is given by:

$$r_t + \gamma V^\pi(s_{t+1}),$$

We can see that using this method, we update our value for $V^\pi(s_t)$ directly after witnessing the transition (s_t, a_t, r_t, s_{t+1}).

Temporal Difference Learning

Algorithm 5: TD Learning to evaluate policy π

Input: step size α, number of trajectories n

For all states s, $V^\pi(s) \leftarrow 0$

while $n > 0$ **do**

    Begin episode E at state s

    while *episode E has not terminated* **do**

        $a \leftarrow$ action at state s under policy π

        Take action a in E and observe reward r, next state s'

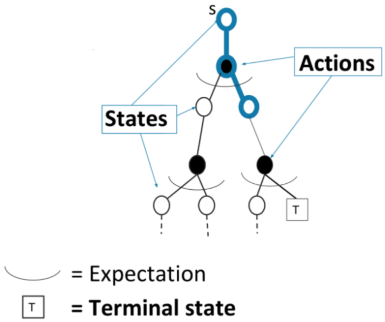
        $V^\pi(s) \leftarrow V^\pi(s) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$

        $s \leftarrow s'$

    $n \leftarrow n - 1$

return V^π

Temporal Difference Learning



Here, we see via the blue line that we sample one transition starting at s, then we estimate the value of the next state via our current estimate of the next state to construct a full Bellman backup estimate.

Temporal Difference Learning

Remark. There is actually an entire spectrum of ways we can blend Monte Carlo and dynamic programming using a method called TD(λ).

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \hat{v}(S_{t+n}, \mathbf{w}_{t+n-1}), \quad 0 \leq t \leq T-n,$$

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{T-t-1} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t-1} G_t,$$

- When $\lambda = 0$, we get the TD learning, hence giving us the alias TD(0).
- When $\lambda = 1$, we recover the Monte-Carlo policy evaluation.
- When $0 < \lambda < 1$, we get a blend of these two methods.

Temporal Difference Learning

Remark. There is actually an entire spectrum of ways we can blend Monte Carlo and dynamic programming using a method called TD(λ).

$$G_{t:t+n} \doteq R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n \hat{v}(S_{t+n}, \mathbf{w}_{t+n-1}), \quad 0 \le t \le T-n,$$

$$G_t^\lambda \quad = \quad (1-\lambda) \sum_{n=1}^{T-t-1} \lambda^{n-1} G_{t:t+n} \quad + \quad \lambda^{T-t-1} G_t,$$

For a more thorough treatment of TD(λ), we refer the interested reader to Sections 7.1 and 12.1-12.5 of *Reinforcement learning: An introduction*.

Batch Monte-Carlo sampling and temporal difference

We consider the batch cases of Monte Carlo and TD(0).

- In the batch case, we are given a batch, or set of histories $h_1, \ldots, h_n$, which we then feed through Monte Carlo or TD(0) many times.
- The only difference from our formulations before is that we only update the value function after each time we process the entire batch.

Batch Monte-Carlo sampling and temporal difference

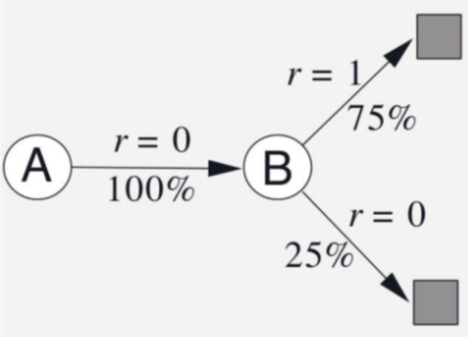
Motivation Example. Suppose $\gamma = 1$ and we have eight histories generated by policy π, take action a_1 in all states:

$$h_1 = (A, a_1, +0, B, a_1, +0, terminal)$$

$$h_j = (B, a_1, +1, terminal) \text{ for } j = 2, \ldots, 7$$

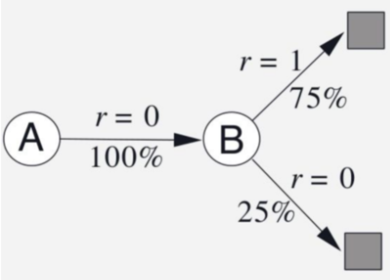
$$h_8 = (B, a_1, +0, terminal).$$

Batch Monte-Carlo sampling and temporal difference



In this example, using either batch Monte Carlo or TD(0) with $\alpha = \frac{1}{N(s)}$, we see that $V(B) = 0.75$.

Batch Monte-Carlo sampling and temporal difference



However, if we use

- Monte Carlo, we get that $V(A) = 0$ since only the first episode visits state A and has return 0.
- TD(0) giving us $V(A) = 0.75$ because we perform the update $V(A) \leftarrow r_{1,1} + \gamma V(B)$. The estimate given by TD(0) makes more sense.

Question and Answering (Q&A)