Lecture 10 - Iterative methods

Guiliang Liu

The Chinese University of Hong Kong, Shenzhen

DDA4230: Reinforcement Learning

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DDA 4230 Resources

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Iterative Policy Evaluation

The iterative policy evaluation algorithm constructs a contraction when $\gamma < 1$, which gives an arbitrarily close value function estimation of a given policy.

- The update $V(s) = \sum_a \pi(a \mid s) \sum_{s',r} \mathbb{P}(s', r \mid s, a)\,[r + \gamma V(s')]$ forms a contraction, such that given V, V', $\|BV - BV'\|_\infty \leq \|V - V'\|_\infty$ where B denotes the operator.

Algorithm 1: Iterative policy evaluation

Input: Policy π, threshold $\epsilon > 0$
Output: Value function estimation $V \approx V^\pi$
Initialize $\Delta > \epsilon$ and V arbitrarily
while $\Delta > \epsilon$ **do**
    $\Delta = 0$
    for $s \in \mathcal{S}$ **do**
        $v = V(s)$
        $V(s) = \sum_a \pi(a \mid s) \sum_{s',r} \mathbb{P}(s', r \mid s, a)\,[r + \gamma V(s')]$
        $\Delta = \max(\Delta, |v - V(s)|)$

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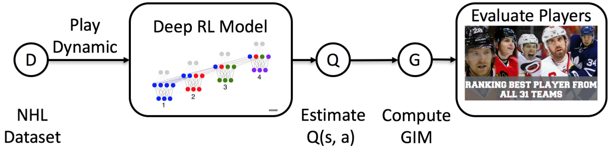
Iterative Policy Evaluation

Application: Player evaluation in Sports Analytics. Players are rated by their observed performance over a set of games. Given dynamic game tracking data, we:

- Apply policy evaluation to estimate the *action value* function $Q(s, a)$, which assigns a value to action a given game state s.
- Compute the player evaluation metric based on the aggregated impact (GIM, i.e., advantages) of their actions over the entire game or season.



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Dynamic programming

For **a finite horizon MDP**, the iterative policy evaluation algorithm requires the iteration to go through the index with a non-stationary value function. This process is known as dynamic programming. By the Bellman equation,

$$
\begin{aligned}
V_t(s) &= R(s) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s, \pi) V_{t+1}(s') \ , \ \forall\, t = 0, \ldots, H-1 \,, \\
V_T(s) &= 0 \,.
\end{aligned} \tag{1}
$$

For episodic MDPs, R and $\mathbb{P}$ can be stochastic and we run this process for many episodes (usually denoted as T/H episodes with horizon H).

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Dynamic programming

Algorithm 2: Iterative policy evaluation with finite horizon

Input: $\mathcal{S}, \mathbb{P}, \mathcal{R}, T$

For all states $s \in \mathcal{S}$, $V_T(s) \leftarrow 0$

$t \leftarrow T - 1$

while $t \geq 0$ **do**

  For all states $s \in S$, $V_t(s) = \sum_a \pi(a \mid s) \sum_{s',r} \mathbb{P}(s', r \mid s, a) \left[r + \gamma V_{t+1}(s')\right]$

  $t \leftarrow t - 1$

return $V_t(s)$ for all $s \in S$ and $t = 0, \ldots, T$

Iterative Policy Search

The policy evaluation algorithm immediately renders itself to a brute force algorithm called policy search to find the optimal value function V^* and an optimal policy π^*.

- The input is an infinite horizon MDP $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \mathbb{P}, \mathcal{R}, \gamma)$ with arbitrary initial state distribution ρ_0 and a tolerance ε for accuracy of policy evaluation,

Algorithm 3: Policy search

Input: $\mathcal{M}, \epsilon$

$\Pi \leftarrow$ All stationary deterministic policies of M

$\pi^* \leftarrow$ Randomly choose a policy $\pi \in \Pi$

$V^* \leftarrow$ POLICY EVALUATION $(\mathcal{M}, \pi^*, \epsilon)$

for $\pi \in \Pi$ **do**

  $V^\pi \leftarrow$ POLICY EVALUATION $(\mathcal{M}, \pi, \epsilon)$

  if $V^\pi(s) \geq V^*(s)\ \forall\ s \in S$ **then**

    $V^* \leftarrow V^\pi$

    $\pi^* \leftarrow \pi$

return $V^*(s),\ \pi^*(s)$ for all $s \in S$

Iterative Policy Search

The policy evaluation algorithm immediately renders itself to a brute force algorithm called policy search to find the optimal value function V^* and an optimal policy π^*.

- The Algorithm terminates as it checks all $|\Pi| = |\mathcal{A}|^{|\mathcal{S}|} = m^n$ deterministic stationary policies (Recall that we are assuming that there exists an optimal policy and in this case there is a deterministic stationary policy that is optimal).
- The run-time complexity of this algorithm is $O(|\mathcal{A}|^{|\mathcal{S}|})$.

Lemma

Policy Search returns the optimal value function and an optimal policy when $\varepsilon = 0$.

Policy Iteration

The policy iteration algorithm applies the Bellman operator, which shows that given any stationary policy π, we can find a deterministic stationary policy that is no worse than the existing policy.

Algorithm 4: Policy improvement

Input: V^π

$\hat{\pi}(s) \leftarrow \underset{a \in \mathcal{A}}{\arg\max} \left[R(s,a) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s,a) V^\pi(s') \right], \forall\, s \in S$

return $\hat{\pi}(s)$ for all $s \in S$

The output of Algorithm 4 is at least as good as the policy π corresponding to the input value function V^π, and represents a greedy attempt to improve the policy.

Algorithm 5: Policy iteration

Input: $\mathcal{M}, \epsilon$

$\pi \leftarrow$ Randomly choose a policy $\pi \in \Pi$

while *true* **do**

  $V^\pi \leftarrow$ POLICY EVALUATION $(\mathcal{M}, \pi, \epsilon)$

  $\pi^* \leftarrow$ POLICY IMPROVEMENT $(\mathcal{M}, V^\pi)$

  if $V^{\pi^*} = V^\pi$ **then**

    break

  else

    $\pi \leftarrow \pi^*$

$V^* \leftarrow V^\pi$

return $V^*(s),\ \pi^*(s)$ for all $s \in S$

Policy Iteration

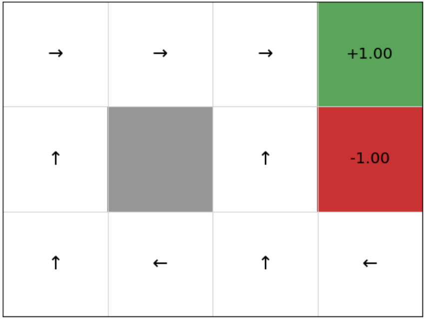
Lemma

Consider an infinite horizon MDP with $\gamma < 1$. The following statements hold.

1. *When Algorithm 5 is run with $\varepsilon = 0$, it finds the optimal value function and an optimal policy.*
2. *If the policy does not change during a policy improvement step, then the policy cannot improve in future iterations.*
3. *The value functions corresponding to the policies in each iteration of the algorithm form a non-decreasing sequence for every $s \in S$.*

Policy Iteration

Policy iteration in Grid World.



Value Iteration

Value Iteration computes the optimal value function and an optimal policy given a known MDP. For every element $U \in \mathbb{R}^n$ the Bellman optimality backup operator B^* is defined as:

$$(B^* U)(s) = \max_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s,a) U(s') \right], \ \forall\, s \in S\,. \tag{1}$$

Value Iteration

Theorem

For a MDP with $\gamma < 1$, let the fixed point of the Bellman optimality backup operator B^* be denoted by $V^* \in \mathbb{R}^n$. Then the policy given by

$$\pi^*(s) = \arg\max_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s,a) V^*(s') \right], \forall\, s \in S \tag{1}$$

will be a stationary deterministic policy. The value function of this policy V^{π^*} satisfies the identity $V^{\pi^*} = V^*$, and V^* is also the fixed point of the operator B^{π^*}.

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Value Iteration

The above theorem suggests a straightforward way to calculate the optimal value function V^* and an optimal policy π^*. The idea is to run fixed point iterations to find the fixed point of B^*. Once we have V^*, an optimal policy π^* can be extracted using the arg max operator in the Bellman optimality equation.

Algorithm 6: Value iteration

Input: ϵ

For all states $s \in S$, $V'(s) \leftarrow 0$, $V(s) \leftarrow \infty$

while $\|V - V'\|_\infty > \epsilon$ **do**

  $V \leftarrow V'$

  For all states $s \in S$, $V'(s) = \max\limits_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s,a) V(s')\right]$

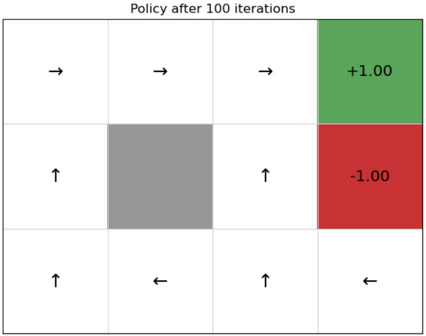
$V^* \leftarrow V$ for all $s \in S$

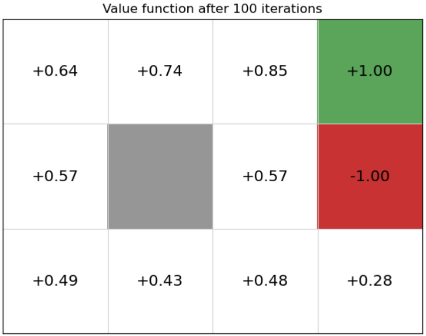
$\pi^* \leftarrow \arg\max\limits_{a \in A} \left[R(s,a) + \gamma \sum_{s' \in S} \mathbb{P}(s' \mid s,a) V^*(s')\right] \quad, \forall\, s \in S$

return $V^*(s)$, $\pi^*(s)$ for all $s \in S$

Value Iteration

Value Iteration in Grid World.





Question and Answering (Q&A)